

## Article

# Behavioral Decision Making in Normative and Descriptive Views: A Critical Review of Literature

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**Abstract:** Recent studies on decision analytics frequently refer to the topic of behavioral decision making (BDM), which focuses on behavioral components of decision analytics. This paper provides a critical review of literature for re-examining the relations between BDM and classical decision theories in both normative and descriptive reviews. We attempt to capture several milestones in theoretical models, elaborate on how the normative and descriptive theories blend into each other, thus motivating the mostly prescriptive models in decision analytics and eventually promoting the theoretical progress of BDM—an emerging and interdisciplinary field. We pay particular attention to the decision under uncertainty, including ambiguity aversion and models. Finally, we discuss the research directions for future studies by underpinning the theoretical linkages of BDM with fast-evolving research areas, including loss aversion, reference dependence, inequality aversion, and models of quasi-maximization mistakes. This paper helps to understand various behavioral biases and psychological factors when making decisions, for example, investment decisions. We expect that the results of this research can inspire studies on BDM and provide proposals for mechanisms for the development of D-TEA (decision—theory, experiments, and applications).

**Keywords:** behavioral decision making; decision analytics; decision theory; behavioral economics; preferences

**JEL Classification:** D81; D91



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## 1. Introduction

There is a long tradition of studying decision making (DM) from normative, descriptive, and prescriptive views. Economists study DM under strong assumptions, such as the well-axiomatized von Neumann and Morgenstern expected utility theory (von Neumann and Morgenstern 1944, vNM hereafter). Once these assumptions are satisfied, people's preferences can be represented by quantitative and measurable values of utility functions, which rationally tell people what decisions to make. However, these strong assumptions, such as (absolute) rationality and self-interests, seem too ideal to describe the reality of people's daily DM. Behavioral economists attempt to amend it through plenty of experimental evidence and theoretical treatments. The Kahneman–Tversky prospect theory (Kahneman and Tversky 1979, PT hereafter) is well accepted as one of the best examples among many other attempts. The PT does not try to guide people to make decisions; instead, it tries to describe the reality of people's decisions more accurately—a descriptive theory. As emphasized by Chai and Ngai (2020), descriptive and normative theories should not be distinguished by the interpretation of the models but by their mathematical forms of models.

Prescriptive theories of DMs lie in between the two sides. It attempts to offer rational and practical recommendations for decision makers. It entails desired normative principles as well as captures actual behaviors of people. Thus, prescriptive decision research has

also been named *the engineering of decisions* and is often studied in management science and engineering. Since it has the potential to bridge the two sides, prescriptive decision analytics have received much attention. For instance, many fields launched their “behavioral” branch by emphasizing their prescriptive (rather than normative or descriptive) power. Taking the operations research (OR) field as an example, we witness the rapidly growing branch of behavioral OR. Some literature attempts to describe the boundaries of behavioral OR and its research themes. [Morton and Fasolo \(2009\)](#), [Becker \(2016\)](#), [Franco et al. \(2021\)](#), and the probably earliest treatment of [von Winterfeldt and Edwards \(1986\)](#) provide excellent reviews of these efforts. However, none of these and other review papers study behavioral DM (BDM) in a jointly normative and descriptive view of economics. Motivated by this gap, this paper traces the evolutions from normative economic theories to descriptive behavioral theories. We aim to reconcile prescriptive decision analytics with state-of-the-art BDM research.

Prescriptive decision analytics normally contain three core problems: ranking, classification (sorting), and choice. Ranking aims to construct an ordinal rank of the objects from the best to the worst. Classification aims to assign objects to the classes. It can be specified as a sorting problem if predefined classes are preference-ordered<sup>1</sup>. Choice aims to identify the best object or select a finite set of the best objects. Therefore, understanding DM’s development can be conducted via two tracks.

On the first track, DM was studied for ranking and classification, usually accompanied by analyses on multiple evaluation criteria with a large scale of data and information. As a well-established research field, multi-criteria decision making (MCDM) has developed many advanced techniques, for example, analytic hierarchy process (AHP), elimination and choice expressing reality (ELECTRE), and the technique for order performance by similarity to ideal solution (TOPSIS). A detailed review of the literature can be found in [Chai et al. \(2013a\)](#) and [Chai and Ngai \(2020\)](#). On the second track, the DM was to study people’s choices. Choice behavior does not rely on massive data or information. Some scholars in data sciences, computer sciences, and information systems often understand ranking and classification as behaviors of machines or systems. Underpinned by normative theories of utility and preference, choice behaviors have been well studied in economics and marketing. Classical choice theories rely on strong assumptions and have self-contained axiomatic systems, whereas recent developments in behavioral economics are rewriting them through incorporating psychological origins.

BDM has emerged as a field spanning decision science, management, economics, finance, applied psychology, computer science ([Hämäläinen et al. 2013](#); [Koszegi 2014](#); [Thaler 2016](#); [Becker 2016](#); [Wong 2021](#)). Some remarkable and latest reviews of the literature on behavioral economics appear. [Wong’s \(2020\)](#) reviews on behavioral economics and finance can be one of the best. This paper provides a comprehensive intersecting surface on theoretical economics models, including utility theories, stochastic dominance, risk measures, several behavioral models, among other statistical and econometric models. In addition, [Wong \(2020\)](#) discusses a wide scope of applications that use various behavioral, statistical, and econometric models. As the departure from past review papers mentioned, our focus is to seize the time clues in which several pieces among numerous theoretical models have directly contributed to BDM in its narrow sense.

In this paper, we explore the theoretical origins of BDM in an interdisciplinary prescriptive through a critical review of literature. The pieces of theoretical models captured in our paper include classical theories of utility and preference, rank-dependent utility, cumulative prospect theory (CPT), ambiguity aversion, and the source method for decision under ambiguity. We expect to bring new insights to the communities of D-TEA (decision—theory, experiments, and applications) and attract more attention to BDM’s theoretical origins and foundations. Specifically, we attempt to figure out how the normative and descriptive theories in economics and decision science blend into each other to motivate the mostly prescriptive models for decision analysis; how they eventually promote the theoretical development of BDM. We point out the theoretical linkage of BDM with three

fast-evolving topics of research: (a) loss aversion and reference dependence, (b) fairness and social preference, and (c) models of quasi-maximization mistakes, which could be promising directions of research in BDM for the future.

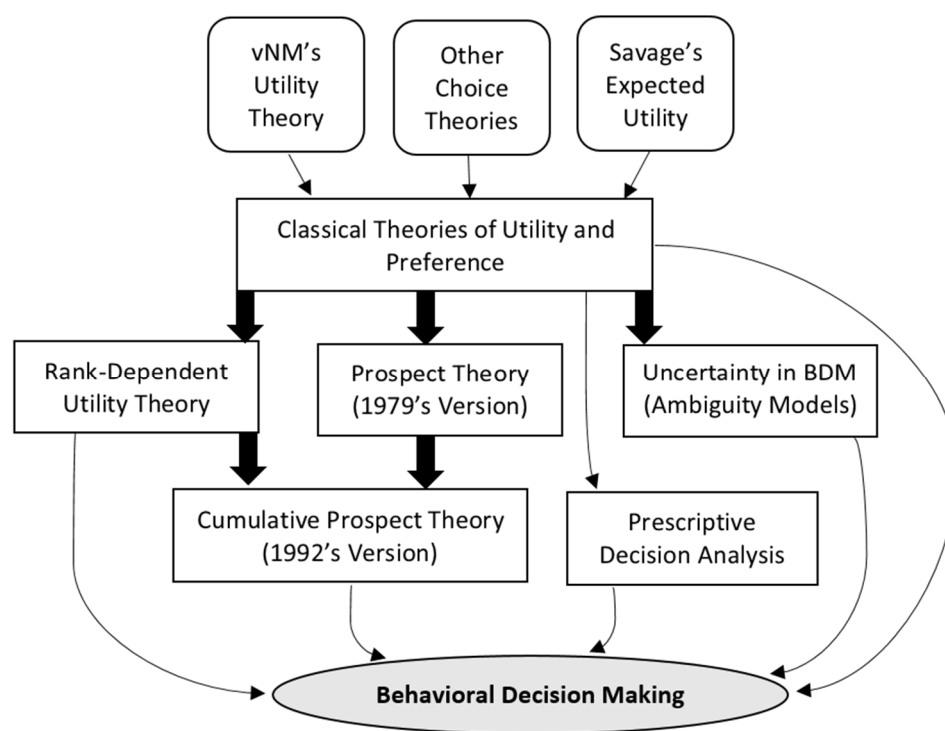
We paid more attention on decision under uncertainty in this paper. Our review is firstly based on normative economic theories and then extended to the descriptive decision analysis. We then focus on the cutting-edge topic on economics' uncertainty—ambiguity aversion and models. It explains home bias phenomena in stock investment (French and Poterba 1991; Fox and Tversky 1995, p. 162; Wakker 2010, p. 280; Wong 2020). That is, people tend to invest more in familiar stocks, rather than new and unfamiliar stocks. These intuitive behaviors in investment decision making can be better understood by ambiguity modeling. Further, we review the source method—a state-of-the-art mechanism—to analyze behavioral bias derived from uncertainty and ambiguity aversion.

We organize this paper as follows. Section 2 states the methodology of this research. Section 3 reviews classical theories of utility and preference. Section 4 reviews PT, including the rank-dependent utility theory as the normative foundation and CPT as the descriptive foundation. Section 5 reviews uncertainties in DM processes. Section 6 reviews ambiguity aversion and Section 7 reviews the source method for handling uncertainties. Section 8 discusses several promising directions for future studies. Section 9 concludes this paper.

## 2. Methodology and Framework

In methodological, we treat BDM as an interdisciplinary field covering management, economics, psychology, and computer science. We select the origins of BDM through retrospectively the historic developments mainly on the theoretical side. Figure 1 illustrates the framework of theoretical pieces that have contributed to the modern BDM. We make Adam Smith's *The Wealth of Nations* as a starting point. Then, important developments include the axioms of revealed preference (WARP, GARP), von Neumann and Morgenstern Expected Utility Theory (EUT, hereafter), Leonard Savage's subjective expected utility (SEU for decision under uncertainty), and so on. They all belong to the classical theories of utility and preferences that have underlain modern decision theories, modern economic theories, and the BDM. Among these, we emphasize the importance of two theoretical branches: the rank-dependence utility theory as a normative representative and the PT as a descriptive representative. Part of these pieces in Figure 1 might be overlapping with other remarkable and comprehensive reviews such as Wong (2020). Our paper is particularly concentrated on the theoretical pieces that we deem as the origins of modern BDM. In addition, our paper also concentrates on uncertainty as it has been the central issue of BDM, which will be addressed in Sections 5–7. For the latest and full treatment on behavioral slices of finance and economics, we recommend Wong (2021) as further reading.

We also present decisions under uncertainty—another important component in DBM. Currently, this issue is centered on ambiguity models or ambiguity aversions. We make Anscombe and Aumann's (1963) model our starting point, a bridge between EUT and SEU. Management scientists are more concerned about the uncertainties in information, environments, and people in the prescriptive decision. The techniques of addressing information uncertainty contain fuzzy set theory (Chen et al. 2021), rough set theory (Chai 2021b), grey system theory (Deng 1982), and so on. Uncertainties studied in economics or prescriptive decision has been one of the origins of modern BDM. In this paper, we will pay particular attention to the decision of uncertainty and helps to exhibit a full picture of the historical and theoretical background of BDM to help readers link their domain knowledge with BDM through this paper and motivate their work in these promising fields.



**Figure 1.** The relational graph of past theoretical branches to the modern BDM.

### 3. Classical Theories of Utility and Preferences

Adam Smith's *The Wealth of Nations* in 1776 was believed as the inception of modern economics. Samuelson (1938) proposed the Weak Axiom of Revealed Preference (WARP), which was recognized as a necessary condition for the maximization of utility. Houthakker (1950) extended WARP as the Generalized Axiom of Revealed Preference (GARP), allowing transitivity in preferences. Its significance is that GARP is a necessary and sufficient condition for utility maximization. von Neumann and Morgenstern (1944, vNM hereafter) delivered a package of axioms containing WARP and the continuity axiom and the independence axiom.

The idea of using the expected value to compute the decision maker's desirability over options was not new. The foundation of probability theory has been proposed from correspondence of Pascal and Fermat in 1654. Daniel Bernoulli's St. Petersburg paradox initially used the concept of "utility" to replace the value, formally,  $EU = pu(x)$ , to improve the flexibility of the model in capturing chooser's attitude over risk. Nevertheless, vNM's work was the first to axiomatize the idea and popularize it. EUT became the heart of modern decision theories till now.

Knight (1921) and Keynes ([1921] 1948) emphasized the necessity of distinguishing a decision under uncertainty (when the probabilities are unknown) from a decision under risk (when the probabilities are known). Motivated by vNM's utility theory, also inspired by Ramsey (1931), a more general formulation of EUT came from Savage's ([1954] 1972) subject expected utility (SEU) theory, which relies on an axiom called "the sure-thing principle". Leonard Savage proved that choosers' preferences should not depend on any "particular" consequence. Although it is essentially a revised version of the independence axiom of EUT, its significance is that the whole EUT model can be extended from risk to uncertainty. This triggered a series of developments of modern decision theory, for example, Anscombe and Aumann (1963), Gilboa and Schmeidler (1989), Schmeidler (1989), and Tversky and Kahneman (1992). Therein, Anscombe and Aumann's (1963) approach attempts to combine the EUT and the SEU theory. Any theory, both in natural and social sciences, should be subject to verification of its falsification. In the history of decision theory, such examples of falsification contain Allais's (1953) paradox and Ellsberg's (1961)

paradox. The former challenged the descriptive validity of vNM's EUT and called its axioms into question. The latter directly attacks Savage's SEU and forcefully claims that it is just a normative rather than a descriptive, theoretical framework.

The gist is as follows. For the Allais paradox, the common ratio effect systematically violates the EUT's independence axiom. Meanwhile, the common consequence effect systematically violates the SEU's sure-thing principle. Intuitively, people are more sensitive to the differences in probability near impossibility ( $p = 0$ ) and near certainty ( $p = 1$ ) than within the intermediate range (e.g.,  $p = 0.5$ ). However, the probability component of the EUT formula fails to characterize it. For the Ellsberg paradox, a two-urn experiment was presented to illustrate that a state's probability will influence people's willingness to bet on the state. They prefer to bet on an event they are more familiar with and are unwilling to bet when there exists uncertainty about probability. Unfortunately, the SEU theory cannot capture this behavior because it assumes that subjective probabilities are always known. This phenomenon is known as "ambiguity aversion," when ambiguity is defined as the uncertainty about probability.

#### 4. Prospect Theory

Colin Camerer and Martin Weber provided critical reviews on SEU in their 1987's article (Weber and Camerer 1987, WC87 hereafter) and on SEU theory in their 1992's article (Camerer and Weber 1992, CW92 hereafter). Related reviews of literature appeared in the 1980s, for example, Fishburn (1988, 1989), Karni and Schmeidler (1990), and Edwards (1992). In the 1980s, the headway in decision theory is perhaps the PT and afterward the CPT by Tversky and Kahneman (1992, TK92 hereafter). Original prospect theory (OPT, hereafter) in its 1979's version was motivated to resolve the fourfold pattern of risk attitude that cannot be captured by vNM's EUT. The scientific community has widely recognized its significance, honored by the 2002 Nobel Prize in Economics awarded to Kahneman. Readers can see a full treatment by Barberis (2013). CPT advances OPT by incorporating all four key characters: (a) loss aversion, (b) reference dependence, (c) diminishing sensitivity, and (d) probability weighting. CPT incorporates OPT and rank-dependent utility (RDU hereafter) as two important ingredients. The main contributors of the RDU theory include Quiggin (1982), e.g., Schmeidler (1989), e.g., Gilboa (1987), e.g., Wakker (1984, 1989), and their colleagues. The primitive target is to extend Savage's SEU model from additive probability to nonadditive probability. CPT intelligently inherits OPT's descriptive ability and RDU's normative ability, and has been the leading model in behavioral economics to date. For a full treatment of CPT from an SEU/RDU's view, one can see Gilboa (2009), Wakker (2010), and Fox and Poldrack (2014).

The key deviation of RDU from classic SEU is a nonadditive probability proposed by Schmeidler (1989) via an axiomatic method on SEU under the AA framework. When SEU's sure-thing principle is weakened to apply only "comonotonic" acts, SEU can be generated from additive probabilities to allow nonadditive probabilities. If so, for the other part in the formula of preference evaluation, utility calculation must also be changed. The solution is to employ Choquet integrals (Choquet 1953). RDU is thus derived as a new deviation of SEU. RDU is also called Choquet utility theory among literature. The model degenerates to SEU easily when the probability of the RDU is additive.

Many researchers further contributed to our understanding of decisions under uncertainty. Wakker (1984) proposed a "cardinal coordinate independence" axiom under the EU framework and adapted this axiom to Schmeidler's comonotonic acts<sup>2</sup>, so that SEU allowed nonadditive probabilities (Wakker 1989). For the same purpose, Gilboa (1987) used a variant of the sure-thing principle restricted to comonotonic acts with an infinite state space. Sarin and Wakker (1992) provided an axiom of using the idea of "cumulative consequence sets" to generate nonadditive probability without comonotonicity. It surprisingly still derives SEU with nonadditivity. The concept of "cumulative dominance" entails that an individual should prefer the act that gives good consequences in more likely states. This concept is a counterpart of the EU's "stochastic dominance" (Hadar and Russell 1971)



that has been extended thereafter in both theoretical and application levels, for example, as in Wong (2007) and Wong and Chan (2008). Together with other axioms, it can control the monotonicity in probability (between  $P(A)$  and  $A$ ) but cannot control additivity in probability. CW92 noted that this concept is only applicable under the SEU framework<sup>3</sup>. Gilboa and Schmeidler (1993) studied how to update nonadditive probabilities and sets of probabilities in generalized SEU approaches—a pseudo-Bayesian updating. Extending SEU with nonadditive probability is important in the explanation of ambiguity-aversion behaviors. Meanwhile, there are other significant developments in the literature concerning the utility component. Readers can see Smith (1969), Karni (1985), and Winkler (1991). Moreover, nonadditive probabilities were addressed in the belief theories of Dempster (1968) and Shafer (1976), and were considered by Luce and his colleagues, including Luce (1991) and Luce and Fishburn (1991). Luce's ideas on decisions under uncertainty were reviewed by Wakker (2000) and also advanced by Chai et al. (2015).

As OPT only allows two non-zero outcomes by incorporating the features of RDU, the version of CPT extends OPT in two aspects: (1) rank dependence and (2) many-outcome lotteries. In CPT, Choquet integrals are used to compute weighted values of consequences. It further separates gain and loss parts in both weighting and utility. CPT now incorporates the features of reference-dependence, sign-dependence, and rank-dependence. The earliest probability theory delivered the expected value model as  $EV = px$ . Underlying Bernoulli's expected utility (EU) model and vNM's axiomatization, the functional formula is  $EU = pu(x)$  concerning a prospect  $x_p0$ . Savage's ([1954] 1972) SEU model is given by  $p(E_j)u(x_j)$ . All these models are sound theoretically. However, some psychological models, for example, the old weighting model with the formula  $w(p_j)x_j$  as well as the OPT model with the formula  $w(p_j)u(x_j)$ , are unsound due to a violation of stochastic dominance (and continuity). One can see detailed arguments in Wakker (2010, p. 153). This imperfection in theory—the state of decision theory in the 1980s—directly resulted in the appearance of rank-dependent models for risk (Quiggin 1982) and for uncertainty (Schmeidler 1989). In this sense, the 1979 version of CPT incorporating the 1980 achievement of RDU was a necessary action. One can review how CPT advances OPT by incorporating historical ingredients of decision theories (i.e., RDU) in Tversky and Kahneman (1992).

In OPT, the value  $v$  of a simple prospect that pay  $\$x$  with probability  $p$  (and nothing otherwise) is given as  $v(x,p) = w(p)u(x)$ . The attractiveness/wantability (i.e., subjective value) of outcome  $x$  is measured by a utility function  $u(\cdot)$ . The impact of probability  $p$  on the attractiveness of the two (non-zero) prospects is measured by weighting function  $w(\cdot)$ . The utility function exhibits the psychophysics of diminishing sensitivity—the marginal impact of a change in value diminishes with the distance from a relevant reference point with  $u(0) = 0$ . Thus, the function is concave for gain and convex for losses. The probability function  $w(\cdot)$  captures diminishing sensitivity to change in probability with two reference points: impossibility ( $p = 0$ ) and certainty ( $p = 1$ ). The utility function is inverse-S-shaped, concave near zero, and convex near one. This OPT model well captures risk attitude, resolves the Allais paradox, and fundamentally improves the descriptive ability of classical EU theory. Two important features of CPT should be highlighted: (1) sign dependence—segregating into gain portions and loss portions in both utility function (as in OPT, but missed by RDU), and weighting function (missed by both OPT and RDU); (2) rank dependence—decision weights is evaluated by cumulative probabilities rather than just one corresponding probability (as the core feature of RDU, but missed by OPT). As a minor feature, CPT is defined concerning the decision under uncertainty. Essentially, it allows CPT to be available for both decision under risk and uncertainty.

## 5. Uncertainty in Economics and Non-Economics

In this section, we capture the uncertainty as a clue to review related literature. Using a lottery-act formulation with a state-dependent representation, Anscombe and Aumann's (1963, AA hereafter) model encompasses both EU and SEU as a special case. The basic formula is as follows. We have a lottery  $L(s) = (x_1, p_1; \dots; x_m, p_m)$  where the outcome  $x_i$  with

its objective probability  $p_i$  for  $i = (1, \dots, m)$ . The outcome of this lottery depends on which state  $s_i$  (concerning its outcome  $x_i$  and probability  $p_i$ ) occurs and on which outcome  $x_i$  the lottery  $L(s_i)$  yields. They called the state-probability representation— $L(s_i)$  and  $p(s_i)$ —as a *horse lottery* and the outcome-probability representation  $(x_i, p_i)$  as a *roulette lottery*. There exist two situations. In the state space, a state  $s_k$  (if occurs) may not have any outcome  $x_k$ , or further indeed having  $x_k$  yielded  $s_k$  but no objective probabilities are available. In this case, a roulette lottery (it requires all objective probability are available, therefore, EU holds) can be regarded as a refined/further stage of a horse lottery<sup>4</sup>. In this sense, Savage's SEU framework is not a deviation of the EU model but generalizes the EU by further considering the situation of missing objective probability.

Sarin and Wakker (1992) advanced this state-dependent representation in the AA model such that the states can be either unambiguous (as roulette lotteries) or ambiguous (as horse lotteries). This essentially unified two stages—first of horse lottery and second of roulette lottery—into one stage. Considering objective/certain probability and subjective/uncertainty probability as two extremes would make sense to understand that the decision model for ambiguity bridged the broader space between EU (objective probability models) and SEU (subjective probability models)<sup>5</sup>.

The AA model is based on EU axioms to represent preferences. The preference is ordered (including completeness and transitivity), continuity, and independence over event-contingent prospects. Under six axioms of Savage's SEU, a unique probability measure (distribution) over all stages, denoted as  $p(s)$ , was defined. In vNM's EU, Savage's SEU, and the AA model, the probability measure must be additive. The more general model of probabilistic sophistication (Machina and Schmeidler 1992, MS92 hereafter) was developed afterward. It suggests that beliefs can be measured by eliciting a matching probability  $P$ , so that over the outcomes each event-contingent prospect can be evaluated by a probability-contingent prospect. As a normative requirement (Machina and Schmeidler 1995), its significance is to relax the formula of preference evaluation by allowing non-expected utility and correlating objective and subjective probability theories. The original version of RDU relaxes the requirement of EU, SEU, or the AA model as to the nonadditive probability, whereas MS92's probability sophistication (PS henceforth) establishes a bridge between objective and subjective probabilities. When RDU meets the requirement of probability sophistication, the modern version of RDU might be the first non-expected utility model for decision under uncertainty (also for risk). It can also be deemed as an advanced model of Savage's SEU, vNM's EU, and the AA model. However, RDU (and EU) as a pure normative theoretical model only can capture subjective probability (and objective probability)—the extreme of ambiguity (of beliefs).

The notion of PS thus has been a bridge from risk to uncertainty under the RDU model. It generalizes subjective probability in the SEU model and allows for more general (e.g., non-expected utility) evaluations over probability distributions. If a chooser is indifferent between receiving  $x_j$  if  $E_j$  occurs and receiving  $x_j$  with probability  $p_j$ , the probability  $p_j$  is a matching probability of  $E_j$ , denoted as  $P(E_j) = p_j$ . Note that  $p_j$  here must be additive and independent of  $x_j$ . The (probability) weighting function  $w$  of  $P(E_j)$  can be rewritten as a new (event-contingent) weighting function  $W$ , for  $W(E_j) = w(P(E_j))$ , with the properties: (i)  $W(\emptyset) = 0$ ; (ii)  $W(S) = 1$  for the universal event  $S$ ; (iii) if  $A \supset B$  then  $W(A) \geq W(B)$ .  $W$  is the composition of  $w$  and  $P$ .  $W$  is nonadditive because  $w$  is nonadditive. This generalizes SEU's additive (subjective) probability.  $W$  does not rely on any probability of events anymore, therefore, eliminating probability sophistication. In modeling ambiguity, e.g., Ellsberg paradox,  $W$  can be interpreted as the notion of *willingness to bet*.

The utility-based theories of decision under uncertainty imply the assumption that uncertainties can be quantified in terms of their probability—a component of the preference evaluation formula. The uncertainties in terms of probability also correlate with other fields as psychology, artificial intelligence, and neuroeconomics (Wakker 2008, p. 436). In artificial intelligence (AI)—a popular topic currently—there exists another notion of uncertainties, which has nothing to do with the probability of outcomes. In this research

line of literature, uncertainties come from information being an intermediary in the decision processes. This information of decision may come from one-side decision makers (e.g., judges or evaluators) or from other-side decision makers (e.g., choosers or selectors or classifiers). In this sense, decision processes might be multi-stage and multi-involvers under different roles. Normally, this process is easy to be implemented by computers. Nevertheless, such information collected in a certain way may not be implementable for computers, for example, if the information is missing, fuzzy, inaccurate, or involves the non-structural perception of people. How to handle these uncertainties can be critical. For a full treatment on this topic, one can see the earlier works conducted by [Chai et al. \(2012, 2013a, 2013b\)](#), [Chai and Liu \(2013, 2014\)](#), and [Chai and Ngai \(2015, 2020\)](#).

Particularly, [Chai et al. \(2013a\)](#) emphasized the key features of this theme in the following three aspects:

- (1) Human preference is represented by (different types of) information. The preference aggregation degenerates as the information aggregation—the intermediate variables.
- (2) Prospects (e.g., lotteries or options) are materialized and termed alternatives.
- (3) Value function in its simplest form is employed for the evaluation of each alternative. The sum of evaluations over all criteria is the index used for ranking, where the evaluation is in the form of  $WV$ , with the value of information  $V$  and weights of criteria  $W$ .

This theme is usually labeled as MCDM. One can see [Wallenius et al. \(2008\)](#), [Marttunen et al. \(2017\)](#), [Cinelli et al. \(2020\)](#) for further reading.

Uncertainty in MCDM is fundamentally different from uncertainty in Savage's SEU, RDU, or other notions of uncertainty addressed in economics. Uncertainties represented by formal probabilities are commonplace in economic theories, probability theory, game theory, and auctions. In MCDM, nevertheless, uncertainty in people's beliefs is usually represented by various types of information, for example, as in [Dempster \(1968\)](#), [Shafer \(1976\)](#), and [Zadeh \(1978\)](#). This has been applied in various disciplines such as information/data sciences, artificial intelligence, pattern recognition. "Soft computing" was the commonly used term for this research line of the literature, for example, [Chai \(2021b\)](#). Arguably, computer scientists often welcome a large scale of data and information because adequate technical tools have been in place to handle "big data" with the assistance of computers. In contrast, economists often tend to evade massive information in analytics. In methodology, the difference in data-driven and model-driven emphasis could be the key to understanding uncertainty between economics and computer science. [Golman et al. \(2016\)](#) provided a systematic review of the literature on information avoidance. [Hansen \(2014\)](#) discusses more general cases.

## 6. Ambiguity Aversion and Models

Apart from Savage's SEU for decision under uncertainty, the richer types of uncertainty exist. This section captures an active branch of uncertainty in decision theory: ambiguity aversion and models. Many authors have made important contributions to explaining ambiguity aversion. The earliest discussion is probably [Keynes's \(\[1921\] 1948\)](#) thought experiment. The most famous work was [Ellsberg's \(1961\)](#) two-urn paradox, which challenges [Savage's \(\[1954\] 1972\)](#) sure-thing principle in SEU. This topic has attracted researchers' attention for more than half a century. We capture the noteworthy literature in ambiguity aversion with three lines of investigations: PT-based ambiguity models, independent (non-PT-based) ambiguity models, and relevant empirical studies.

### (1) PT-based ambiguity models:

Under the PT paradigm, uncertainty derives from different sources, and attitude toward ambiguity depends on the source of uncertainty. The pioneering work in this area is known as the "support theory" proposed by [Tversky and Koehler \(1994\)](#) and [Rottenstreich and Tversky \(1997\)](#). Subsequent studies include [Tversky and Wakker \(1995\)](#) and [Kilka and Weber \(2001\)](#). [Fox and Tversky \(1998\)](#) found that people prefer to bet on their vague beliefs than on matched chance events, called a competence hypothesis. People first assess the



probability  $p$  of an uncertain event  $A$ ; then transform this value using a risky weighting function  $w$ . The process is thus deemed as a two-stage model.

Abdellaoui et al. (2011, ABPW11) proposed a purely descriptive model that concerns the source of uncertainty based on CPT. This method uses three components to trace decisions under uncertainty, including the utility of outcomes, the choice-based probabilities, and the new add-on, source functions. It, in fact, separates ambiguity attribute (over sources) and beliefs (i.e., probabilities). It uses a source function to capture ambiguity attitudes, rather than modifying beliefs (i.e., probability component) and taste (i.e., utility component) as conducted in the past. Wakker (2008) summarized models of uncertainty into four categories: rank-dependence, multi-priors, “model-free” approaches, and finally multi-stage approaches, along with their representations. The source method is based on the first categories (i.e., rank dependence) to detect the source of uncertainty.

### (2) Non-PT-based independent ambiguity models:

The independent models to explain ambiguity aversion include Choquet expected utility (EU) model (Schmeidler 1989), maxmin EU (Gilboa and Schmeidler 1989),  $\alpha$ -maxmin EU (Ghirardato et al. 2004), variational preference model (Maccheroni et al. 2006), smooth ambiguity model (Klibanoff et al. 2005), vector EU model (Siniscalchi 2009). For more independent ambiguity models, one can see Gajdos et al. (2008), Ergin and Gul (2009), Cerreia-Vioglio et al. (2011), Hansen and Sargent (2001), and Hansen et al. (1999).

### (3) The noteworthy empirical studies

Ambiguity aversion has been empirically studied in Halevy (2007) and Machina (2009). Since the separation of decision under risk and uncertainty, two tracks are developed independently. It led to Savage’s SEU theory being the well-known departure of classical vNM’s EUT. Fortunately, risk can be considered as a special case of uncertainty where probability  $p_j$  is given for the event  $E_j$ , thus  $W(E_j) = w(p_j)$ . The probability  $p_j$  is called the matching probability of  $E_j$ , denoted as  $p_j = P(E_j)$  if indifference hold between receiving  $x_j$  if  $E_j$  occurs and receiving  $x_j$  with  $p_j$ . Under the SEU framework, MS92 proposed the PS model. It is regarded as a normative requirement as MS95. The matching probability  $p_j$ , a tool to measure beliefs, must be additive and independent of outcome  $x$ . That matching probability  $P_j$  is independent of the sign; gain and loss is not a must.

## 7. The Source Method in Decision under Ambiguity

A source (of uncertainty) is a group of events, which is generated by a common mechanism of uncertainty, for example the urns in the Ellsberg paradox. A source  $S$  is called *uniform* if PS holds over  $S$ . The counterpart includes “revealed equal likelihood” and exchangeability. Chew and Sagi (2008) initially proposed the concept of “homogeneity”. If  $x_{E1}y \sim x_{E2}y$ , we say that the events  $E_1$  and  $E_2$  are “revealed equal likelihood”, denoted as  $E_1 \sim E_2$ . Exchangeability is a stronger condition over events and over a group of events.  $E_1$  and  $E_2$  are exchangeable if exchanging the outcomes under them do not influence the preference for a prospect  $(E_1:x_1, E_2:x_2, \dots, E_n:x_n)$ . A partition  $(E_1, \dots, E_n)$  is exchangeable if all events are mutually exchangeable. (Savage [1954] 1972) termed partitions with exchangeability as “uniform”. Two different (event-contingent) prospects are equivalent in terms of preference relation if they can generate the same probability distribution over outcomes; thus, PS holds, given  $p_j = P(E_j)$  for all  $j$ .

The source method introduced by ABPW11 recommends using source functions to analyze uncertainty. Events coming from the same source have common features and are generated by a common mechanism in terms of uncertainty. Therefore, it is reasonable to analyze uncertainty in choosing with consideration to the source of events. Inspired by Ellsberg’s paradox, PS is usually violated between sources and often still satisfied within a single source. In this sense, the source method essentially applies PS as a benchmark: if PS holds, a source  $S$  is called *uniform*. One can see Chew and Sagi (2006, 2008) for further reading.

Wakker (2008) believed PS is a uniform degree of ambiguity for that source, leading to the ABPW11’s source method. Wakker (2008, p. 433) wrote, “The Ellsberg paradox entails

a comparison of attitudes of one agent concerning different sources of uncertainty." An example by him is used to illustrate ambiguity over different "sources". Let source be the algebra of events. Source A regards the Dow Jones stock index tomorrow, whereas source B regards the Japanese Nikkei stock index. Suppose event  $a$  from A and event  $b$  from B; it means that the indexes will go up tomorrow over  $a$  and  $b$ . If one prefers  $100_a0$  to  $100_b0$ , then it also suggests that one may be influenced by a preference over "source", A over B, further this phenomenon was caused by "source A comprises less ambiguity for this chooser than source B does" (e.g., this chooser is an American who knows more about her local stock market). Or simply, it is because event  $a$  is more likely to occur than  $b$ . How to measure such differences in "chooser-perceived levels of likelihood", or formally how to capture choosers' ambiguity attitudes, is the core motivation of relevant research. Proposed solutions include [Ergin and Gul \(2009\)](#) and [Chew and Sagi \(2006\)](#).

[Wakker's \(2008\)](#) idea is to find a partition over A and B in which events entails  $100_a0$  is more preferred to  $100_b0$  for all  $j$ , then define a new preference (depend on different sources) but beyond belief (i.e., subjective probability over outcomes). The tool to perform so is probability sophistication (PS). PS implies that a common probability-contingent prospect can represent two event-contingent prospects, say,  $100_a0 \sim 100_b0 \sim 100_p0$  where  $p_j = P(a_j) = P(b_j)$ ;  $p_j$  is called matching probability under PS, must be additive (because  $P(\cdot)$  does), and independent of the outcome (e.g., 100). If PS holds in the partition, the source is indifferent; events within are revealed equal likelihood (captured by  $p_j$ ) and thus exchangeable (also called a uniform source). A chooser is said to be probabilistically sophisticated if individuals' choice behavior can be measured by probabilistic beliefs. However, the rest of the partition (non-PS), which reflects the chooser's ambiguity attitude over source will be captured independently beyond beliefs. With all these in mind, PS essentially entails a *uniform degree of a source*, reflects how ambiguity is derived from the difference of source (not from the events). This part of ambiguity (depend on the source) can be captured by measuring the degree of deviation from PS over a source. Based on this idea, [ABPW11](#) proposed the concept of source function  $Ws(\cdot)$ , replacing the event-contingent weighting function in RDU  $W(\cdot)$ , and descriptively depicted its graphs. They proposed two indexes to characterize the graphs, essentially to capture the degree of PS-deviation, they are: (a) the degree of pessimism (optimism) and (b) the degree of likelihood insensitivity.

## 8. Discussion

The rationality assumed in economics seems to be too strict for capturing people's actual decisions. The idea of "bounded-rationality" frequently used in behavioral economists seems to be a good compromise, which has helped the BDM penetrate market, management, organization, information systems, even accounting, and finance, as demonstrated in [Becker \(2016\)](#). Classical theories of utility and preference contain powerful tools to normatively explain individual choice behaviors, although the capacity of explanations is limited in certain situations. Incorporating the descriptive view from psychological findings into strictly normative economic theories has proven fruitful. Further, the combination concatenates the engineering of decision that strongly relies on prescriptive decision analytics. Thus, BDM as a research theme came to people's view eventually.

Daniel Kahneman and the well-known PT (joint with Tversky Amos) as we reviewed before must be one of the most important "people and things" in improving the psychological realm of economics. As emphasized, this "psychology and economics" theory also motivates the descriptive potential of the prescriptive decision analytics and further promotes the formation of modern BDM. Currently, behavioral economics research has become explosive. Basic theoretical frames have penetrated many fields. Among many achievements, three theoretical models are promising for future research:

- (1) Loss aversion and reference dependence, e.g., Kahneman and Tversky's PT;
- (2) Fairness and social preference, e.g., [Fehr and Schmidt's \(1999\)](#) fairness model;
- (3) Models of quasi-maximization mistakes, e.g., Rabin, Laibson, and their coauthors.

The so-called “big three” regularities in behavioral modeling have been widely recognized and accepted by other disciplines, for example, marketing (Ho et al. 2006) and organizational behavior (Koszegi 2014). Related discussions can be found in Kahneman and Tversky (2000), Rabin (1998, 2013), and Thaler (2016).

An outstanding problem in modern BDM is incorporating the “big three” regularities into information and data science. Based on the extensive literature review in Chai et al. (2013a) and Chai and Ngai (2020), incorporating economic models into decision system analysis is seldom used before. Recently, Chai (2021a) proposed a utility-based happiness model that captures the two-fold effect of self-adaptations and interpersonal comparisons in evaluating time streams of payoffs. This theoretical framework incorporates Fehr and Schmidt’s fairness model and a modification of the classical discounted utility model to measure and predict human behaviors through subjective happiness as the medium. In the future, how to incorporate behavioral principles of loss aversion, reference dependence, fairness (inequity aversion), social preference, and quasi-maximization mistakes to resolve the problems of decision processes can be a promising direction of BDM.

## 9. Conclusions

As one can see from our account, the modern BDM is a highly interdisciplinary field. Through a critical review of literature, this paper reveals that the BDM has been deeply rooted in long-standing, well-established normative economic theories, and the psychology-oriented descriptive theoretical frameworks. We grasped the “normative” theories of utility and preferences, the “normative” rank-dependent utility theory, the “descriptive” PT, the “normative” theories in the decision under uncertainty, the “prescriptive” decision analysis, and their strategies on handling the uncertainty of information. Our reviews and analyses in economics’ uncertainty, ambiguity aversion, and the source method provide proposals of understanding behavioral bias and psychological factors in investment decisions. In summary, we have exhibited normative, descriptive, and prescriptive theoretical backgrounds of modern BDM studies. The modern BDM has been the best intersection among management, economics, psychology, informatics, and data sciences, for which research opportunities abound in the future.

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## Notes

- <sup>1</sup> When the classes are not predefined, such classification is called a clustering or grouping problem, which is an important problem in the field of data mining and machine learning. Interested readers can refer to Han et al. (2012).
- <sup>2</sup> For state  $a$  and  $b$ , acts  $f$  and  $g$  are called “comonotonic” if  $f(a) \succ f(b)$  implies  $g(a) \succeq \bigcap g(b)$ , thus state  $a$  and  $b$  can be (weakly) ranked. Wakker (2010, p. 279 and Appendix 10.12) commented that the notion of “comonotonicity” proposed by Schmeidler (1989) is not intuitive for applications.
- <sup>3</sup> Under nonadditive probabilities, Gilboa and Schmeidler (1989) pointed out that there exist some curious problems, which still adhere to utility maximization.
- <sup>4</sup> All states have the sure outcomes, but their probabilities can be subjective rather than objective, therefore, SEU holds.

- <sup>5</sup> Wakker (2010) pointed out that such a misunderstanding, say Savage's SEU model is to deviate the objective probability models such as EU theory, is primarily in psychological literature. It, unfortunately, leads to a bigger misunderstanding in modern decision theory as the separation between risk and uncertainty. If so, all modern decision models under risk shall have a generalized version for uncertainty, and all models for uncertainty might also be suitable for risk in one of its degenerated versions.

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